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One-dimensional partially asymmetric simple exclusion process with open boundaries: orthogonal polynomials approach

Tomohiro Sasamoto

Department of Physics, Graduate School of Science, University of Tokyo, Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan

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Abstract. The stationary state of the partially asymmetric simple exclusion process with open boundaries is reconsidered. The so-called matrix product ansatz is employed. This enables us to construct the stationary state in the form of the product of the matrices D and E. Noticing the fact that the matrix $C(= D+E)$ for the model is related to certain q-orthogonal polynomials, the model is analysed for a wide range of parameters. The current and the correlation length are evaluated in the thermodynamic limit. It turns out that the phase diagram for the correlation length is richer than that for the totally asymmetric case.

1. Introduction

The stationary properties of one-dimensional lattice gases are currently of much research interest [1]. The systems are known to show rich non-equilibrium behaviours. For instance, they exhibit boundary-induced phase transitions [2], spontaneous symmetry breaking [3] and phase separation [4, 5]. The one-dimensional asymmetric simple exclusion process (ASEP) [6, 7] is one such system and has been studied extensively. The ASEP is defined as follows: particles hop on a one-dimensional lattice with an exclusion rule that prevents more than one particle occupying the same site. During the infinitesimal time interval $\mathrm{d}t$, each particle jumps to the right nearest-neighbouring site with probability $p_R\,\mathrm{d}t$ and to the left nearest-neighbouring site with probability $p_L\mathrm{d}t$. If the chosen site is already occupied, the jump is suppressed due to the exclusion rule. More than one particle cannot be on the same site. The case where particles can hop only in one direction, i.e., the case where either $p_L = 0$ or $p_R = 0$ is called the 'totally asymmetric' case. Another extreme case $p_L = p_R$ will be called the 'symmetric' case in the sense that particles hop symmetrically to the right and left. Lastly, the remaining case, where particles hop in both directions with different rates, will be referred to as the 'partially asymmetric' case.

The stationary state of the ASEP depends on the boundary condition in an essential way. The periodic boundary condition corresponds to the system on a ring. The stationary state is rather trivial. All possible configurations have equal probability and the density profile is flat. The ASEP with the reflective boundary condition was studied in [8]. The stationary state is described by the representation theory of the quantum algebra $U_q[SU(2)]$. The density shows a shock-like profile. The most interesting is, however, the case where we employ open boundary conditions. We allow the particle input at the left end of the chain with rate α and

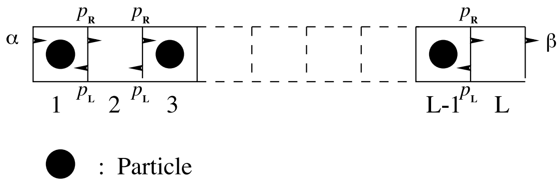

Figure 1. The one-dimensional asymmetric simple exclusion process with open boundaries.

allow the particle output at the right end of the chain with rate β (figure 1). Here the length of the chain is denoted by L. The model is then known to exhibit phase transitions depending on the values of the parameters α and β. For the totally asymmetric case, this fact was first noticed in [2] by numerical simulations. It was then confirmed by exact calculation using the recursion relation in [9–11].

The so-called matrix product ansatz was originally introduced in [12] to study a two-dimensional directed animals. It was then applied to the ASEP in [13] to reproduce and generalize the results in [9, 10]. This enables us to construct the stationary state of the ASEP in terms of two matrices, which satisfy certain algebraic relations. In [13], the method was applied to the totally asymmetric case. Since then, techniques have been generalized and successfully applied to many other interesting problems [14, 15] such as the calculations of the diffusion constant [16–18], the formation of shocks [19–24], discrete time dynamics [25–31], reaction–diffusion models [32], models with disorder [33] and multi-species cases [34–37]. These applications have been mainly for the totally asymmetric case. There are several exact results for the symmetric case [38] and for the partially asymmetric case as well [13,18,22,23,39–41]. However, the partially asymmetric case with the open boundary conditions has not been fully exploited. Known facts are summarized as follows. First, an example of infinite-dimensional representations of the algebraic relations was given in [13]. Second, the algebraic relation was related to the so-called q-boson in [40]. The current was calculated by employing some plausible approximations. Third, there exist finite-dimensional representations for special choices of the parameters [39, 41]. Then the current and the density profile are calculated rather easily.

In this paper, the partially asymmetric case is considered for a wide range of parameters. The restrictions on the parameters are $0 \leqslant p_L \leqslant p_R$ and $\alpha, \beta > 0$. We note that the process has an obvious particle–hole symmetry. When we look at holes instead of particles, they tend to hop to the left with rate p_R and to the right with rate p_L with hard-core exclusion. In addition, they are injected at the right end with rate β and removed at the left end with rate α. In other words, the process is invariant under the changes,

$$\begin{aligned} &\text{particle} \leftrightarrow \text{hole} \\ &\alpha \leftrightarrow \beta \\ &\text{site number } j \leftrightarrow \text{site number } L - j + 1. \end{aligned} \tag{1.1}$$

Due to this symmetry, it is sufficient to consider the case where $\alpha \leqslant \beta$. To evaluate the physical quantities, we use the fact that a matrix appearing in this problem is intimately related to the Al-Salam–Chihara polynomials [42]. These polynomials are some of the so-called q-orthogonal polynomials and can be written in the form of basic hypergeometric series [43]. The Al-Salam–Chihara polynomials are a special case of the Askey–Wilson polynomials, which were first introduced in [44]. It should be noticed that the Askey–Wilson polynomials are the general classical orthogonal polynomials which contain many important orthogonal polynomials as special cases.

The paper is organized as follows. In the next section, the construction of the stationary state by means of the matrix product ansatz is briefly reviewed. In section 3, some notations for the q-calculus are introduced. They are extensively used in the following discussions. In section 4, the q-boson and the q-Hermite polynomials are introduced to study the algebraic relations for the matrices. The special case $\alpha = \beta = p_R - p_L$ is studied in section 5. The current is represented in the form of the integral and is evaluated asymptotically by using the steepest-descent method. For this special case, the explicit formula for finite L is also found. To obtain the results for more general case, the Al-Salam–Chihara polynomials are introduced in section 6. In section 7, the current is evaluated in the thermodynamic limit. The phase diagram for the current is recovered. In section 8, the correlation length is calculated. The phase diagram for the correlation length is identified. It turns out to have a richer structure than that for the totally asymmetric case. In section 9, some known special cases are discussed from the viewpoint of the orthogonal polynomials. The last section is devoted to the concluding remarks.

2. Matrix product ansatz

It is sometimes useful to formulate the stochastic processes in the language of quantum mechanics [45]. This enables us to analyse many interesting processes by the techniques originally devised for studying quantum mechanical systems. This is based on a simple observation that time evolution for the stochastic systems can be described by the master equation,

$$\frac{\mathrm{d}}{\mathrm{d}t} P(t) = -HP(t) \tag{2.1}$$

which is formally the same as the (imaginary-time) Schrödinger equation. Here H is a transition rate matrix of the process and $P(t)$ represents the probability distribution of the system. Though the operator H is in general non-Hermitian, it will be called a Hamiltonian hereafter.

The Hamiltonian for the ASEP with the open boundary condition has a form,

$$H = h_1 + \sum_{j=1}^{L-1} h_{j,j+1} + h_L. \tag{2.2}$$

Here the matrix $h_{j,j+1}$ $(1 \leqslant j \leqslant L-1)$ acts as a 4×4 matrix h on the jth and $(j+1)$th spaces and as an identity on the other spaces. The matrix h represents the asymmetric diffusion and is explicitly given by

$$h = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & p_L & -p_R & 0 \\ 0 & -p_L & p_R & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.3}$$

On the other hand, h_1 (resp. h_L) acts non-trivially only on the 1st (resp. Lth) space and corresponds to the particle input and output at the left (resp. right) end of the lattice:

$$h_1 = \begin{bmatrix} \alpha & 0 \\ -\alpha & 0 \end{bmatrix} \qquad h_L = \begin{bmatrix} 0 & -\beta \\ 0 & \beta \end{bmatrix}. \tag{2.4}$$

Let us now briefly review the matrix product ansatz [13]. In the matrix product ansatz, the stationary state of the ASEP is assumed to be of the form,

$$P = \langle W| \begin{pmatrix} E \\ D \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} E \\ D \end{pmatrix} |V\rangle / Z_L \tag{2.5}$$

where Z_L is the normalization constant given below. Here D and E are square matrices acting in an auxiliary space. The auxiliary space is not specified at this stage. The dimension of this space depends on the values of the parameters. There are both finite-dimensional representations and infinite-dimensional representations. The matrix product ansatz is based on a cancellation mechanism of the local interaction described by the matrices h, h_1 and h_L. One can see that the stationary state condition $HP = 0$ is solved if the matrices D, E and the vectors $\langle W|$, $|V\rangle$ satisfy the following relations:

$$h\begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} = \zeta \left\{ \begin{pmatrix} \bar{E} \\ \bar{D} \end{pmatrix} \otimes \begin{pmatrix} E \\ D \end{pmatrix} - \begin{pmatrix} E \\ D \end{pmatrix} \otimes \begin{pmatrix} \bar{E} \\ \bar{D} \end{pmatrix} \right\} \tag{2.6a}$$

$$\langle W|h_1\begin{pmatrix} E \\ D \end{pmatrix} = -\zeta \langle W|\begin{pmatrix} \bar{E} \\ \bar{D} \end{pmatrix} \qquad h_L\begin{pmatrix} E \\ D \end{pmatrix}|V\rangle = \zeta \begin{pmatrix} \bar{E} \\ \bar{D} \end{pmatrix}|V\rangle \tag{2.6b}$$

where $\bar{D}$ and $\bar{E}$ are some square matrices and ζ is an arbitrary number. In the case of ASEP, we can set $\bar{E} = -\bar{D} = 1$ and the relations are reduced to

$$p_R DE - p_L ED = \zeta(D + E) \tag{2.7a}$$

$$\alpha\langle W|E = \zeta\langle W| \qquad \beta D|V\rangle = \zeta|V\rangle. \tag{2.7b}$$

Once one finds a representation of these algebraic relations, one can in principle calculate the physical quantities such as the particle current J_L, the one-point function $\langle n_j\rangle_L$, the two-point function $\langle n_j n_k\rangle_L$ and the higher correlation functions. Here the subscript L stands for the lattice length. In the subsequent discussions, the matrix C defined by

$$C = D + E \tag{2.8}$$

plays an important role. In terms of the matrix C, the normalization Z_L is given by

$$Z_L = \langle W|C^L|V\rangle. \tag{2.9}$$

The one-point function $\langle n_j\rangle_L$ is defined as the probability that the site j is occupied. In other words, $\langle n_j\rangle_L$ is the average density at site j. The two-point function $\langle n_j n_k\rangle_L$ is defined as the probability that the sites j and the site k are both occupied. Higher-correlation functions are defined similarly. They are computed by the formula,

$$\langle n_j\rangle_L = \langle W|C^{j-1}DC^{L-j}|V\rangle/Z_L \tag{2.10}$$

$$\langle n_j n_k\rangle_L = \langle W|C^{j-1}DC^{k-j-1}DC^{L-k}|V\rangle/Z_L \tag{2.11}$$

and so on. The current through the bond between site j and site $j+1$ is defined by $J_L^{(j)} = p_R\langle n_j(1-n_{j+1})\rangle - p_L\langle n_j(1-n_{j-1})\rangle$. In the steady state, the current is independent of j and hence is denoted by J_L. In the matrix representation, it is given by

$$J_L = \zeta\frac{\langle W|C^{L-1}|V\rangle}{\langle W|C^L|V\rangle} = \zeta\frac{Z_{L-1}}{Z_L}. \tag{2.12}$$

The totally asymmetric case was solved in [13] by using the above matrix formulation. The current and the one-point function were computed. On the other hand, the exact results for the partially asymmetric case are less known. For special values of the parameters, the matrices have finite-dimensional representations and the physical quantities are readily computable [39, 41]. However, for the other range of parameters, we have to employ an infinite-dimensional representation. So far, when the matrices are infinite-dimensional, only the current was computed with some approximations [40].

3. Some notations for q-calculus

Before going to the main discussions, several notations for the so-called q-calculus are introduced [43,46]. They will be used extensively in the following sections. First we introduce the q-number,

$$\{n\} = 1 - q^n \tag{3.1}$$

and the q-factorial,

$$\{n\}! = \{n\}\{n-1\}\ldots\{1\} \tag{3.2}$$

for $n = 0, 1, 2, \ldots$. If we take appropriate limits, they reduce to the usual number and the factorial as

$$\lim_{q\to 1} \frac{\{n\}}{1-q} = n \qquad \lim_{q\to 1} \frac{\{n\}!}{(1-q)^n} = n!. \tag{3.3}$$

Second, we define

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j) \tag{3.4}$$

for $|q| < 1$. The infinite product in (3.4) converges when $|q| < 1$ for all $a \in \mathbb{C}$. We also define the q-shifted factorial,

$$\begin{aligned}(a;q)_n &= \frac{(a;q)_\infty}{(aq^n;q)_\infty} \\ &= (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{n-1}) && (3.5a)\\ (a;q)_0 &= 1. && (3.5b)\end{aligned}$$

Here the condition $|q| < 1$ is unnecessary. If we set a to be q^a and take $q \to 1$, it reduces to the shifted factorial,

$$\lim_{q\to 1} \frac{(q^a;q)_n}{(1-q)^n} = \prod_{j=0}^{n-1}(a+j). \tag{3.6}$$

Since products of q-shifted factorials appear so often, we use the notations,

$$(a_1, a_2, \ldots, a_k; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty \ldots (a_k;q)_\infty \tag{3.7}$$

$$(a_1, a_2, \ldots, a_k; q)_n = (a_1;q)_n (a_2;q)_n \ldots (a_k;q)_n. \tag{3.8}$$

Lastly, the basic hypergeometric series (or q-hypergeometric series) is defined by the series

$${}_r\phi_s\left(\begin{matrix}a_1, a_2, \ldots, a_r\\ b_1, b_2, \ldots, b_s\end{matrix}; q, z\right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_s; q)_n} ((-)^n q^{n(n-1)/2})^{1+s-r} \frac{z^n}{(q;q)_n}. \tag{3.9}$$

It is assumed that the parameters $b_1, b_2, \ldots, b_s$ are such that the denominator factors in the terms of the series are never zero. When $0 < |q| < 1$, the ${}_r\phi_s$ series converges absolutely for all $z \in \mathbb{C}$ if $r \leqslant s$ and for $|z| < 1$ if $r = s + 1$. The basic hypergeometric series tends to the usual hypergeometric series as $q \to 1$.

4. q-boson and q-Hermite polynomials

In [40], it was pointed out that the algebraic relation (2.7a) is related to the so-called q-boson [47–50]. The q-boson is defined by the three relations among the three operators B, $B^\dagger$ and N,

$$BB^\dagger - qB^\dagger B = 1 - q \tag{4.1a}$$

$$[N, B^\dagger] = B^\dagger \qquad [N, B] = -B. \tag{4.1b}$$

The q-boson appears in various physical situations. In the context of the stochastic processes, it should be noted that there is an asymmetric diffusion process the Hamiltonian of which is written in terms of the q-boson [51]. Just like the ordinary boson, the q-boson has the following Fock representation:

$$B^\dagger|n\rangle = \{n+1\}^{1/2}|n+1\rangle \tag{4.2}$$

$$B|n\rangle = \{n\}^{1/2}|n-1\rangle \tag{4.3}$$

$$N|n\rangle = n|n\rangle. \tag{4.4}$$

Here the Fock basis is denoted by $|n\rangle$ with $n = 0, 1, 2, \ldots$. In addition, the q-boson has the coherent state. It is called the q-coherent state and is defined by

$$|\lambda\rangle_c = \sum_{n=0}^{\infty} \frac{\lambda^n}{\{n\}!}(B^\dagger)^n|0\rangle = \sum_{n=0}^{\infty} \frac{\lambda^n}{(\{n\}!)^{\frac{1}{2}}}|n\rangle. \tag{4.5}$$

It satisfies

$$B|\lambda\rangle_c = \lambda|\lambda\rangle_c \qquad {}_c\langle\lambda|B^\dagger = \lambda\, {}_c\langle\lambda|. \tag{4.6}$$

To see the relationship between the algebraic relation (2.7a) and the q-boson, take $\zeta = p_R - p_L$ and set

$$D = 1 + d \qquad E = 1 + e \tag{4.7}$$

in (2.7a) and (2.7b). Here and hereafter we will assume $p_R \neq p_L \neq 0$ except in sections 9.2 and 9.3. Then we see that these relations become

$$de - qed = 1 - q \tag{4.8a}$$

$$\langle W|e = a\langle W| \qquad d|V\rangle = b|V\rangle \tag{4.8b}$$

where we put

$$q = p_L/p_R \tag{4.9}$$

$$a = \frac{1-\tilde{\alpha}}{\tilde{\alpha}} \qquad b = \frac{1-\tilde{\beta}}{\tilde{\beta}} \tag{4.10}$$

with $\tilde{\alpha} = \alpha/(p_R - p_L)$, $\tilde{\beta} = \beta/(p_R - p_L)$. Since $0 < p_R < p_L$, $\alpha > 0$ and $\beta > 0$, we have $0 < q < 1$, $a > -1$ and $b > -1$. As will become clear, the parameters a and b are more fundamental than the original parameters α and β. Now we see that relation (4.8a) is nothing but one of the defining relations of the q-boson (see (4.1a)). Hence one can take d and e to be the Fock representations of the operators B and $B^\dagger$, respectively. They are denoted by d_1 and e_1. In matrix notation, they are explicitly given by

$$d_1 = \begin{bmatrix} 0 & \{1\}^{\frac{1}{2}} & 0 & 0 & \cdots \\ 0 & 0 & \{2\}^{\frac{1}{2}} & 0 & \\ 0 & 0 & 0 & \{3\}^{\frac{1}{2}} & \\ \vdots & & & \ddots & \ddots \end{bmatrix} \qquad e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ \{1\}^{\frac{1}{2}} & 0 & 0 & 0 & \\ 0 & \{2\}^{\frac{1}{2}} & 0 & 0 & \\ 0 & 0 & \{3\}^{\frac{1}{2}} & 0 & \\ \vdots & & & \ddots & \ddots \end{bmatrix}. \tag{4.11}$$

If we take this representation, relations (4.8*b*) show that the vector $\langle W|$ (resp. $|V\rangle$) is the left (resp. right) eigenvector of the operator $e_1 = B^\dagger$ (resp. $d_1 = B$) with the eigenvalue a (resp. b). Hence $\langle W|$ and $|V\rangle$ can be taken as the q-coherent states (4.5). They are denoted by $\langle W_1|$ and $|V_1\rangle$:

$$\langle W_1| = \kappa\, {}_c\langle a| \qquad |V_1\rangle = \kappa |b\rangle_c \tag{4.12}$$

with $\kappa^2 = (ab; q)_\infty$. The normalization κ is taken so that $\langle W|V\rangle = 1$.

Next we define the matrix,

$$\begin{aligned} T_1 &= d_1 + e_1 \\ &= \begin{bmatrix} 0 & \{1\}^{\frac{1}{2}} & 0 & 0 & \cdots \\ \{1\}^{\frac{1}{2}} & 0 & \{2\}^{\frac{1}{2}} & 0 & \\ 0 & \{2\}^{\frac{1}{2}} & 0 & \{3\}^{\frac{1}{2}} & \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}. \end{aligned} \tag{4.13}$$

We notice that the matrix T_1 is a real, symmetric tridiagonal matrix with positive off-diagonal entries. Such matrices are called Jacobi matrices. They are known to be closely related to the theory of orthogonal polynomials [52, 53]. In addition, the matrix (4.13) is bounded. Associated with each bounded Jacobi matrix T, there exists a set of orthogonal polynomials $\{p_n(x)|\, n = 0, 1, 2, \ldots\}$ which are orthogonal with respect to a probability measure with compact support. Defining $|p(x)\rangle = {}^t(p_0(x), p_1(x), \ldots)$, we require that the orthogonal polynomials satisfy

$$T|p(x)\rangle = 2x|p(x)\rangle \tag{4.14}$$

with the initial condition,

$$p_{-1}(x) = 0 \qquad p_0(x) = 1. \tag{4.15}$$

In other words, the vector $|p(x)\rangle$ is formally an eigenvector of the Jacobi matrix T with the eigenvalue $2x$. It should be noted that equation (4.14) is equivalent to impose the three term recurrence relation on $\{p_n(x)\}$ with initial condition (4.15). Fortunately, the weight function is known for the Jacobi matrix T_1 in (4.13). The three-term recurrence relation of the orthogonal polynomials $\{p_n(x)\}$ for the Jacobi matrix T_1 reads

$$\{n\}^{1/2} p_{n-1}(x) + \{n+1\}^{1/2} p_{n+1}(x) = 2x p_n(x). \tag{4.16}$$

Setting

$$P_n(x) = (\{n\}!)^{1/2} p_n(x) = \sqrt{(q;q)_n}\, p_n(x) \tag{4.17}$$

the recurrence relation becomes

$$P_{n+1}(x) + (1-q^n) P_{n-1}(x) = 2x P_n(x). \tag{4.18}$$

This is exactly the three-term recurrence relation of the continuous q-Hermite polynomials $\{H_n(x|q)|n = 0, 1, 2, \ldots\}$. Hence we have $P_n(x) = H_n(x|q)$. The continuous q-Hermite polynomials were first introduced in [54] and have been studied extensively [55, 56].

We list some properties of the polynomials which we will need in our calculations. See, for instance, [53]. The continuous q-Hermite polynomials are explicitly given by the formula,

$$H_n(x|q) = \sum_{k=0}^{n} \frac{\{n\}!}{\{k\}!\{n-k\}!} \mathrm{e}^{\mathrm{i}(n-2k)\theta} \tag{4.19}$$

with $x = \cos\theta$. The orthogonality relation of them reads

$$\int_{-1}^{1} H_n(x|q) H_m(x|q) w(x) (1-x^2)^{-1/2}\, \mathrm{d}x = 2\pi \frac{(q;q)_n}{(q;q)_\infty} \delta_{mn}. \tag{4.20}$$

Here the weight function $w(x)$ is

$$w(x) = \prod_{k=0}^{\infty}(1 - 2(2x^2 - 1)q^k + q^{2k}) \\ = (\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}; q)_\infty \tag{4.21}$$

where again $x = \cos\theta$. The generating function is also known and is given by

$$\sum_{n=0}^{\infty} \frac{H_n(\cos\theta|q)}{\{n\}!}\lambda^n = \frac{1}{(\lambda\mathrm{e}^{\mathrm{i}\theta}, \lambda\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty} \tag{4.22}$$

for $|\lambda| < 1$. Finally, the completeness of the continuous q-Hermite polynomials can be written as

$$1 = \frac{(q;q)_\infty}{2\pi}\int_0^\pi \mathrm{d}\theta\, w(\cos\theta)|p(\cos\theta)\rangle\langle p(\cos\theta)| \tag{4.23}$$

where the function $w(x)$ is given by (4.21).

5. The case $\alpha = \beta = p_R - p_L$

In this section, the case where $a = b = 0$ is considered. In terms of the original parameters, this case corresponds to the case $\alpha = \beta = p_R - p_L$. This case is especially easy to analyse because the vectors $\langle W_1|$ and $|V_1\rangle$ in (4.12) reduce to

$$\langle W_1| = {}_c\langle 0| = \langle 0| = (1, 0, 0, \ldots) \qquad |V_1\rangle = |0\rangle_c = |0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{5.1}$$

The simplicity of this special case was also noted in [39]. Now we represent the normalization Z_L for this case in the form of an integral. Since the vector $|p(\cos\theta)\rangle$ is an eigenvector of the matrix T_1 with eigenvalue $2\cos\theta$, one sees

$$C_1^L|p(\cos\theta) = [2(1+\cos\theta)]^L|p(\cos\theta)\rangle \tag{5.2}$$

where we defined $C_1 = T_1 + 2$. Noticing

$$\langle 0|p(\cos\theta)\rangle = 1 \qquad \langle p(\cos\theta)|0\rangle = 1 \tag{5.3}$$

we calculate the normalization Z_L as

$$\begin{aligned} Z_L &= \langle 0|C_1^L|0\rangle \\ &= \frac{(q;q)_\infty}{2\pi}\int_0^\pi \mathrm{d}\theta\, w(\cos\theta)\langle 0|C_1^L|p(\cos\theta)\rangle\langle p(\cos\theta)|0\rangle \\ &= \frac{(q;q)_\infty}{2\pi}\int_0^\pi \mathrm{d}\theta\, w(\cos\theta)[2(1+\cos\theta)]^L\langle 0|p(\cos\theta)\rangle\langle p(\cos\theta)|0\rangle \\ &= \frac{(q;q)_\infty}{2\pi}\int_0^\pi \mathrm{d}\theta\, w(\cos\theta)[2(1+\cos\theta))]^L \end{aligned} \tag{5.4}$$

where the function $w(x)$ is given by (4.21).

Using the steepest-descent method, we can evaluate this integral in the limit $L \to \infty$ as follows. First we rewrite the function $w(x)$ as

$$w(\cos\theta) = 4\sin^2\theta f(\cos\theta) \tag{5.5}$$

with $f(\cos\theta) = (q\mathrm{e}^{2\mathrm{i}\theta}, q\mathrm{e}^{-2\mathrm{i}\theta}; q)_\infty$. Making a change of variable,

$$\mathrm{e}^{-u} = \frac{1+\cos\theta}{2} \tag{5.6}$$

(5.4) is rewritten as

$$Z_L = \frac{2\cdot 4^{L+1}(q;q)_\infty}{\pi}\int_0^\infty \mathrm{d}u\, \mathrm{e}^{-uL-\frac{3}{2}u}\sqrt{1-\mathrm{e}^{-u}}\, f(2\mathrm{e}^{-u}-1). \tag{5.7}$$

Rescaling the variable u by $v = uL$, we have

$$Z_L = \frac{2\cdot 4^{L+1}(q;q)_\infty}{\pi L^{3/2}}\int_0^\infty \mathrm{d}v\, v^{1/2}\mathrm{e}^{-v}\mathrm{e}^{-\frac{3v}{2L}}\left(\frac{1-\mathrm{e}^{-v/L}}{v/L}\right)^{1/2} f(2^{-v/L}-1). \tag{5.8}$$

Since the function $f(2^{-v/L}-1)$ is continuous and bounded, we can take the limit $L\to\infty$ in the integrand in (5.8). Noticing

$$\lim_{L\to\infty}\mathrm{e}^{-\frac{3v}{2L}}\left(\frac{1-\mathrm{e}^{-v/L}}{v/L}\right)^{1/2} f(2^{-v/L}-1) = (q;q)_\infty^2 \tag{5.9}$$

we finally obtain

$$Z_L \simeq \frac{(q;q)_\infty^3 4^{L+1}}{\sqrt{\pi}L^{\frac{3}{2}}}. \tag{5.10}$$

Let J denote the current in the thermodynamic limit, i.e., $J = \lim_{L\to\infty} J_L$. Using (2.12), the current is easily computed as $J = (p_R - p_L)/4$.

For the $a = b = 0$ case, the explicit formula of Z_L can also be obtained for finite L. We use the formula,

$$\int_0^\pi \mathrm{e}^{2\mathrm{i}k\theta} w(\cos\theta)\,\mathrm{d}\theta = \frac{\pi(-)^k(q^{\frac{1}{2}k(k+1)} + q^{\frac{1}{2}k(k-1)})}{(q;q)_\infty}. \tag{5.11}$$

The proof of this formula can be found in [55]. Noticing a simple fact,

$$[2(1+\cos\theta)]^L = \sum_{k=0}^{2L}\binom{2L}{k}\mathrm{e}^{\mathrm{i}(L-k)\theta} \tag{5.12}$$

we obtain

$$Z_{2l} = \frac{1}{2}\sum_{k=0}^{2l}(-)^{l-k}\binom{4l}{2k}[q^{\frac{1}{2}(l-k)(l-k-1)} + q^{\frac{1}{2}(l-k)(l-k+1)}] \tag{5.13a}$$

$$Z_{2l+1} = \frac{1}{2}\sum_{k=0}^{2l}(-)^{l-k}\binom{4l+2}{2k+1}[q^{\frac{1}{2}(l-k)(l-k-1)} + q^{\frac{1}{2}(l-k)(l-k+1)}] \tag{5.13b}$$

for $l = 0, 1, 2, \ldots$. Here $\binom{m}{n} = m!/(n!(m-n)!)$ is the usual binomial coefficient. This formula is exact for any finite lattice length L. However, the integral formula (5.4) seems more appropriate to obtain the asymptotic behaviour of the quantity.

6. Al-Salam–Chihara polynomials

Next we consider the case where $a = b = 0$ does not necessarily hold. The vectors $\langle W|, |V\rangle$ are now taken to be (4.12) instead of (5.1). For $|a| < 1$, $|b| < 1$, the formula (5.4) is generalized by replacing (5.3) with

$$_c\langle a|p(\cos\theta)\rangle = \frac{1}{(a\mathrm{e}^{\mathrm{i}\theta}, a\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty} \qquad \langle p(\cos\theta)|b\rangle_c = \frac{1}{(b\mathrm{e}^{\mathrm{i}\theta}, b\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty} \tag{6.1}$$

which are simple consequences of (4.22). Hence we have

$$
\begin{aligned}
Z_L &= \langle W_1|C_1^L|V_1\rangle \\
&= \kappa_c^2 \langle a|(T_1+2)^L|b\rangle_c \\
&= \frac{(q;q)_\infty (ab;q)_\infty}{2\pi}\int_0^\pi \mathrm{d}\theta\, w(\cos\theta)[2(1+\cos\theta)]^L\, {}_c\langle a|p(\cos\theta)\rangle\langle p(\cos\theta)|b\rangle_c \\
&= \frac{(q,ab;q)_\infty}{2\pi}\int_0^\pi \mathrm{d}\theta\, w_{a,b}(\cos\theta)[2(1+\cos\theta))]^L
\end{aligned}
\tag{6.2}
$$

where

$$
w_{a,b}(\cos\theta) = \frac{(\mathrm{e}^{2\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta}; q)_\infty}{(a\mathrm{e}^{\mathrm{i}\theta}, a\mathrm{e}^{-\mathrm{i}\theta}, b\mathrm{e}^{\mathrm{i}\theta}, b\mathrm{e}^{-\mathrm{i}\theta}; q)_\infty}. \tag{6.3}
$$

This is the generalization of the expression for the totally asymmetric case (see (B10) in [13]). We can also consider the case where $|a| < 1$, $|b| < 1$ does not hold by using the analytic continuation of the integral as explained in appendix B of [13] for the totally asymmetric case.

As far as only the normalization Z_L is concerned, formula (6.2) together with the analytic continuation is enough. To obtain more detailed information, however, it is better to introduce another set of orthogonal polynomials, which are orthogonal with respect to the function $w_{a,b}(\cos\theta)$ when $|a| < 1$ and $|b| < 1$. The polynomials are known as the Al-Salam–Chihara polynomials [42, 57]. They are represented in q-hypergeometric series as

$$
P_n(a,b;x) = (ab;q)_n a^{-n}\, {}_3\phi_2\left(\begin{matrix} q^{-n}, a\mathrm{e}^{\mathrm{i}\theta}, a\mathrm{e}^{-\mathrm{i}\theta} \\ ab, 0 \end{matrix}; q, q\right) \tag{6.4}
$$

where $x = \cos\theta$. Since

$$
(a\mathrm{e}^{\mathrm{i}\theta}, a\mathrm{e}^{-\mathrm{i}\theta}; q)_k = \prod_{j=0}^{k-1}(1 - 2aq^j\cos\theta + a^2 q^{2j}) \tag{6.5}
$$

it is clear that $P_n(a,b;x)$ is a polynomial of degree n in $x = \cos\theta$. They are a special case of the Askey–Wilson polynomials, which were introduced in [44]. The Askey–Wilson polynomials contain four parameters other than q. Al-Salam–Chihara polynomials correspond to setting two parameters to zero. The Askey–Wilson polynomials have played an important role in the theory of the q-orthogonal polynomials since they contain various important q-orthogonal polynomials such as the q-ultraspherical polynomials and the q-Jacobi polynomials as special cases. In the following, several properties of the Al-Salam–Chihara polynomials are explained. The proofs will not be provided because the properties of the Al-Salam–Chihara polynomials below are obtained as special cases of the properties of the Askey–Wilson polynomials, for which a standard reference is available [44].

The Al-Salam–Chihara polynomials (6.4) are known to satisfy the three-term recurrence relation,

$$
P_{n+1}(a,b;x) + (a+b)q^n P_n(a,b;x) + (1-q^n)(1-abq^{n-1})P_{n-1}(a,b;x) = 2xP_n(a,b;x) \tag{6.6}
$$

with the initial condition $P_{-1}(a,b;x) = 0$ and $P_0(a,b;x) = 1$. In section 4, it was pointed out that a set of orthogonal polynomials is associated with a Jacobi matrix. So one may wonder whether there is a matrix representation of the algebraic relations (2.7a) and (2.7b) associated with the Al-Salam–Chihara polynomials. It turns out that the representation given in appendix A of [13] is directly related to these polynomials. The matrices d, e and the vectors $\langle W|, |V\rangle$ corresponding to this representation will be denoted by d_2, e_2, $\langle W_2|$ and $|V_2\rangle$. Explicitly, they are given by

$$d_2 = \begin{bmatrix} b & \sqrt{c_1} & 0 & 0 & \cdots \\ 0 & bq & \sqrt{c_2} & 0 & \\ 0 & 0 & bq^2 & \sqrt{c_3} & \\ \vdots & & & \ddots & \ddots \end{bmatrix} \qquad e_2 = \begin{bmatrix} a & 0 & 0 & 0 & \cdots \\ \sqrt{c_1} & aq & 0 & 0 & \\ 0 & \sqrt{c_2} & aq^3 & 0 & \\ 0 & 0 & \sqrt{c_3} & aq^4 & \\ \vdots & & & \ddots & \ddots \end{bmatrix} \tag{6.7a}$$

$$\langle W_2| = (1, 0, 0, \ldots) \qquad |V_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{6.7b}$$

where

$$c_n = (1 - q^n)(1 - abq^{n-1}). \tag{6.8}$$

As in section 4, we define the matrix,

$$\begin{aligned} T_2 &= d_2 + e_2 \\ &= \begin{bmatrix} a+b & \sqrt{c_1} & 0 & 0 & \cdots \\ \sqrt{c_1} & (a+b)q & \sqrt{c_2} & 0 & \\ 0 & \sqrt{c_2} & (a+b)q^3 & \sqrt{c_3} & \\ 0 & 0 & \sqrt{c_3} & (a+b)q^4 & \ddots \\ \vdots & & & \ddots & \ddots \end{bmatrix}. \end{aligned} \tag{6.9}$$

The matrix T_2 is again a Jacobi matrix when $ab < 1$. When $ab > 1$, some of the c_n are negative. Then the matrix T_2 is not a Jacobi matrix. However, as we will see, this fact does not cause any difficulty for our discussions. We can define orthogonal polynomials $|p(a, b; x)\rangle = {}^t(p_0(a, b; x), p_1(a, b; x), \ldots)$ which satisfy $T_2|p(a, b; x)\rangle = 2x|p(a, b; x)\rangle$. Setting

$$\begin{aligned} \tilde{P}_n(a, b; x) &= (c_1 c_2 \ldots c_n)^{1/2} p_n(a, b; x) \\ &= \sqrt{(q, ab; q)_n}\, p_n(a, b; x) \end{aligned} \tag{6.10}$$

the polynomials $\tilde{P}_n(a, b; x)$ are shown to satisfy (6.6). Hence we have $\tilde{P}_n(a, b; x) = P_n(a, b; x)$. Actually, we could have started from the representation (4.11), (4.12) instead of the representation (6.7*a*), (6.7*b*). Both representations lead to the same results. The advantage of the representation (6.7*a*), (6.7*b*) is that the dependences of the physical quantities on the parameters a, b can be discussed by only looking at the spectrum of the matrix T_2.

Now we explain the orthogonality relation of the Al-Salam–Chihara polynomials and the spectrum of the matrix T_2. Though the orthogonality relation of the Al-Salam–Chihara polynomials depends on the values of a and b, it can be represented in a single expression in the form of the contour integral as

$$\frac{1}{2\pi \mathrm{i}} \int_C \frac{\mathrm{d}z}{z} \frac{(z^2, z^{-2}; q)_\infty P_n(a, b; (z + z^{-1})/2) P_m(a, b; (z + z^{-1})/2)}{(az, a/z, bz, b/z; q)_\infty} = 2\delta_{m,n} h_n \tag{6.11}$$

where

$$h_n = \frac{(q, ab; q)_n}{(q, ab; q)_\infty}. \tag{6.12}$$

In (6.11) the contour C is the unit circle traversed in the positive direction, but with suitable deformations to separate the sequences of poles converging to zero from the sequences of poles

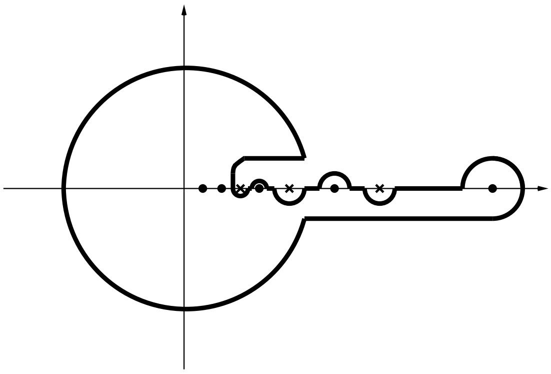

Figure 2. An example of the contour in the integral (6.11). In this example, $0 < b < a^{-1} < 1 < a < b^{-1}$ is assumed. The sequences of poles at $z = aq^j$ ($j = 0, 1, 2, \ldots$) converge to zero and are indicated by filled circles. The sequences of poles at $z = (aq^j)^{-1}$ ($j = 0, 1, 2, \ldots$) diverge to infinity and are indicated by crosses. The sequences of poles at $z = bq^j$ and at $z = (aq^j)^{-1}$ ($j = 0, 1, 2, \ldots$) are not indicated in this figure. The contour is such that it encircles only the sequences of poles converging to zero.

diverging to infinity. In other words, the contour includes all poles of the type cq^n and excludes all poles of the type $1/(cq^n)$ with $c = a, b$. An example of the contour is shown in figure 2. Depicted is the case where $0 < b < a^{-1} < 1 < a < b^{-1}$ holds. In this orthogonality relation, we assume $ab \neq 1, q^{-1}, q^{-2}, \ldots$ except in section 9.1. There, the condition $ab = q^{-n+1}$ for some $n = 1, 2, \ldots$ will be related to the existence of a finite-dimensional representation of the algebraic relations (2.7*a*) and (2.7*b*). The proof of the orthogonality relation (6.11) with (6.12) for the general case is highly non-trivial. An interested reader should consult [44].

For $|a| < 1, |b| < 1$, the contour is nothing but the unit circle and the orthogonality relation for the polynomials is given by

$$\frac{1}{2\pi}\int_0^\pi \mathrm{d}\theta\, w_{a,b}(\cos\theta) P_n(a, b; \cos\theta) P_m(a, b; \cos\theta) = \delta_{m,n} h_n. \tag{6.13}$$

Here the weight function $w_{a,b}(x)$ is (6.3). The orthogonality relation (6.13) means that the spectrum of the matrix T_2 for this case consists only of the continuous spectrum covering $[-2, 2]$. Especially, the largest eigenvalue of the matrix T_2 for this case is two. As a mathematical statement, this sentence is not correct because the value two is not a discrete eigenvalue but in the continuous spectrum. But we abuse the word 'eigenvalue' as in the above sentence unless the meanings are unclear. The completeness of the Al-Salam–Chihara polynomials for this case can be written as

$$1 = \frac{(q, ab; q)_\infty}{2\pi}\int_0^\pi \mathrm{d}\theta\, w_{a,b}(\cos\theta) |p(a, b; \cos\theta)\rangle\langle p(a, b; \cos\theta)| \tag{6.14}$$

where the function $w_{a,b}(x)$ is given by (6.3).

Next we consider the case where $a > 1$ and $|b| < 1$. Since $a > 1$ and $0 < q < 1$, there exists a non-negative integer n such that

$$a > aq > \cdots > aq^n > 1 > aq^{n+1} > \cdots. \tag{6.15}$$

The orthogonality relation for this case reads

$$\frac{1}{2\pi}\int_0^\pi \mathrm{d}\theta\, w_{a,b}(\cos\theta) P_n(a, b; \cos\theta) P_m(a, b; \cos\theta)$$

$$+\sum_{j=0}^{n} P_n(a,b;x_j^{(a)})P_m(a,b;x_j^{(a)})w_j^{(a)} = \delta_{m,n}h_n \tag{6.16}$$

where $x_j^{(a)} = [aq^j + (aq^j)^{-1}]/2$ and $w_j^{(a)}$ is given by

$$w_j^{(a)} = \frac{(a^{-2};q)_\infty (a^2,ab;q)_j (1-a^2q^{2j})}{(q,ab,b/a;q)_\infty (q,aq/b;q)_j (1-a^2) q^{j^2} a^{3j} b^j}. \tag{6.17}$$

Here the summation term in (6.16) comes from the contributions from the poles in the integrand in (6.11) at $z = aq^j$ and $z = (aq^j)^{-1}$ for $j = 0, 1, \ldots, n$. More precisely, one gets the expression (6.17) by considering the residue at the poles of the integrand at $z = aq^j$ and $z = (aq^j)^{-1}$. This expression is also obtained by taking an appropriate limit in the corresponding expression for the Askey–Wilson polynomials [44]. When $ab > 1$, $w_j^{(a)}$ can be negative. The positivity of the weight function is lost. This might be a great disadvantage from the viewpoint of the theory of the orthogonal polynomials. However, this fact has no importance for our discussions. The spectrum of the matrix T_2 consists of the continuous one covering $[-2, 2]$ and the discrete one with the eigenvalues $\{2x_j^{(a)} | j = 0, 1, 2, \ldots, n\}$. Notice that all of the eigenvalues in the discrete spectrum are off $[-2, 2]$. Moreover, we can see that $x_0^{(a)} > x_0^{(a)} > \cdots > x_n^{(a)}$. Especially, the largest eigenvalue of the matrix T_2 for this case is $2x_0^{(a)} = a + a^{-1}$. Finally, the completeness of the Al-Salam–Chihara polynomials for this case reads

$$1 = \frac{(q,ab;q)_\infty}{2\pi} \int_0^\pi d\theta\, w_{a,b}(\cos\theta) |p(a,b;\cos\theta)\rangle\langle p(a,b;\cos\theta)| + \sum_{j=0}^{n} w_j^{(a)} |p(a,b;x_j^{(a)})\rangle\langle p(a,b;x_j^{(a)})|. \tag{6.18}$$

The case where $|a| < 1$ and $b > 1$ is essentially the same as the case where $a > 1$ and $|b| < 1$ with the roles of a and b interchanged. The spectrum of the matrix T_2 consists of the continuous one covering $[-2, 2]$ and the discrete one with the eigenvalues $\{2x_j^{(b)} = bq^j + (bq^j)^{-1} | j = 0, 1, 2, \ldots, n\}$. It is also easy to guess the orthogonality relation when a and b are both larger than one. If $a > aq > \cdots > aq^{n^{(a)}} > 1 > aq^{n^{(a)}+1} > \cdots$ and $b > bq > \cdots > bq^{n^{(b)}} > 1 > bq^{n^{(b)}+1} > \cdots$, there appear two summation terms of the form $\sum_{j=0}^{n^{(c)}} P_n(a,b;x_j^{(c)})P_m(a,b;x_j^{(c)})w_j^{(c)}$ with $c = a, b$ in addition to the integral in (6.13). The spectrum of the matrix T_2 consists of the continuous one ranging $[-2, 2]$ and two discrete series of the eigenvalues $\{2x_j^{(a)} | j = 0, 1, 2, \ldots, n^{(a)}\}$ and $\{2x_j^{(b)} | j = 0, 1, 2, \ldots, n^{(b)}\}$. The largest eigenvalue of the matrix T_2 depends on the values of a and b. When a (resp. b) is larger than b (resp. a), it is given by $a + a^{-1}$ (resp. $b + b^{-1}$).

7. Calculation of Z_L and current

As for the orthogonality relation (6.11), the normalization Z_L for general case is represented as the contour integral:

$$Z_L = \frac{(q,ab;q)_\infty}{4\pi i} \int_C \frac{dz}{z} \frac{(z^2, z^{-2};q)_\infty [(1+z)(1+z^{-1})]^L}{(az, a/z, bz, b/z;q)_\infty}. \tag{7.1}$$

Here the contour is the same as for the orthogonality relation (6.11). Formula (7.1) is obtained as the analytic continuation of (6.2). When $|a| < 1$ and $|b| < 1$, the contour is the unit circle and (7.1) reduces to the integral expression in (6.2). Applying the steepest-descent method as in section 5, we can get the asymptotic expression of Z_L for this case. For the case where

$a > 1$ and $|b| < 1$, there appear other contributions in addition to the integral. When (6.15) holds, Z_L is calculated as

$$Z_L = \frac{(q, ab; q)_\infty}{2\pi} \int_0^\pi \mathrm{d}\theta\, w_{a,b}(\cos\theta)[2(1+\cos\theta))]^L \\ +(q, ab; q)_\infty \sum_{j=1}^{n} [(1+aq^j)(1+(aq^j)^{-1})]^L w_j^{(a)}. \tag{7.2}$$

It turns out that as $L \to \infty$ the contribution of the $j = 0$ term in the summation is dominant. Hence the asymptotic expression for Z_L for this case is

$$Z_L \simeq \frac{(a^{-2}; q)_\infty}{(b/a; q)_\infty}[(1+a)(1+a^{-1})]^L. \tag{7.3}$$

Other cases can be calculated similarly. Due to the particle–hole symmetry, it suffices to know the cases where $a \geqslant b$, or equivalently, the cases where $\tilde{\alpha} \leqslant \tilde{\beta}$. The asymptotic expressions for Z_L are summarized as follows:

- For $\tilde{\alpha} > \frac{1}{2}$ and $\tilde{\beta} > \frac{1}{2}$ $(|a|, |b| < 1)$

$$Z_L \simeq \frac{(ab; q)_\infty (q; q)_\infty^3 4^{L+1}}{\sqrt{\pi}(a, b; q)_\infty^2 L^{\frac{3}{2}}}. \tag{7.4}$$

- For $\frac{1}{2} < \tilde{\alpha} = \tilde{\beta}$ $(|a| = |b| < 1)$

$$Z_L \simeq \frac{(a^2; q)_\infty (q; q)_\infty^3 4^{L+1}}{\sqrt{\pi}(a; q)_\infty^4 L^{\frac{3}{2}}}. \tag{7.5}$$

- For $\tilde{\alpha} = \frac{1}{2} < \tilde{\beta}$ $(a = 1 > |b|)$

$$Z_L \simeq \frac{2 \cdot 4^L}{\sqrt{\pi}(b; q)_\infty L^{\frac{1}{2}}}. \tag{7.6}$$

- For $\tilde{\alpha} < \frac{1}{2}$ and $\tilde{\alpha} < \tilde{\beta}$ $(a > 1, a > b)$

$$Z_L \simeq \frac{(a^{-2}; q)_\infty}{(b/a; q)_\infty}[(1+a)(1+a^{-1})]^L. \tag{7.7}$$

- For $\tilde{\alpha} = \tilde{\beta} < \frac{1}{2}$ $(a = b > 1)$

$$Z_L \simeq \frac{(a - a^{-1})(a^{-2}; q)_\infty L}{(q; q)_\infty}[(1+a)(1+a^{-1})]^{L-1}. \tag{7.8}$$

These are the generalizations of the results for the totally asymmetric case (see (51)–(56) in [13]).

Using formula (2.12), the current is readily calculated in the thermodynamic limit. The expression for the current is different in three regions of the parameter space $(\tilde{\alpha}, \tilde{\beta})$. The results are:

- Case A (low-density phase; $\tilde{\alpha} < \frac{1}{2}$ and $\tilde{\beta} > \tilde{\alpha}$; $a > 1$ and $a > b$)

$$J = (p_R - p_L)\tilde{\alpha}(1 - \tilde{\alpha}). \tag{7.9}$$

- Case B (high-density phase; $\tilde{\beta} < \frac{1}{2}$ and $\tilde{\alpha} > \tilde{\beta}$; $b > 1$ and $a < b$)

$$J = (p_R - p_L)\tilde{\beta}(1 - \tilde{\beta}). \tag{7.10}$$

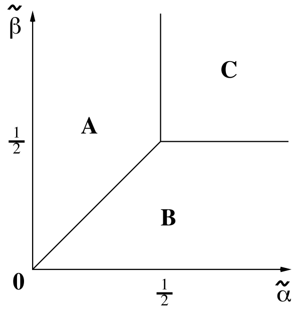

Figure 3. The phase diagram of the current. Regions A, B and C are called the low-density phase, the high-density phase and the maximal current phase, respectively.

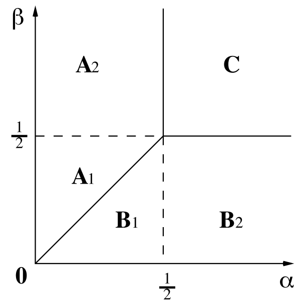

Figure 4. The phase diagram of the correlation length in the α–β plane for the totally asymmetric case. The low-density phase (resp. high-density phase) is divided into two phases, A_1 and A_2 (resp. B_1 and B_2) along the line $\beta = \frac{1}{2}$ (resp. $\alpha = \frac{1}{2}$).

- Case C (maximal current phase; $\tilde{\alpha} > \frac{1}{2}$ and $\tilde{\beta} > \frac{1}{2}$; $|a|, |b| < 1$)

$$J = \frac{p_R - p_L}{4}. \tag{7.11}$$

Hence the phase diagram for the current (figure 3) is recovered. It consists of three phases. Following [9, 11] we refer to cases A, B and C as the low-density phase, the high-density phase and the maximal current phase, respectively. This phase diagram for the current is correctly predicted by the mean-field approximation [39] and was confirmed in [40] with some approximations employed.

8. Correlation length

It is known that the density profile in different phases (A, B and C) have different appearances. This fact is correctly predicted by the mean-field analysis [9,39]. In the low-density phase, the density takes a constant value $\tilde{\alpha}$ in the bulk. It decays exponentially near one boundary and is constant near the other. Similarly, in the high-density phase, the density is constant $1 - \tilde{\beta}$ in the bulk and decays exponentially near one of the boundaries. In the maximal current phase, it is constant in the bulk but it shows power-law decays near both the boundaries. Since the average density is between zero and one, we assume $0 \leqslant \tilde{\alpha}, \tilde{\beta} \leqslant 1$ (i.e. $a, b \geqslant 0$) hereafter. When the density decays exponentially near the boundary as $e^{-r/\xi}$ with r the distance from the boundary, we call ξ the correlation length in the following.

For the totally asymmetric case ($p_R = 1, p_L = 0$), the density profile was exactly calculated in [11, 13]. It turned out that, whereas the mean-field analysis predicts the bulk density correctly, it fails to give the correct correlation lengths for the high- and low-density phases. Actually, the low-density (resp. high-density) phase is divided into two phases A_1 and A_2 (resp. B_1 and B_2) along the curve $\beta = \frac{1}{2}$ (resp. $\alpha = \frac{1}{2}$). See figure 4. The correlation lengths in phases A_1 and A_2 are given by $\xi^{-1} = \ln[\beta(1-\beta)/\alpha(1-\alpha)]$ and $\xi^{-1} = -\ln 4\alpha(1-\alpha)$, respectively. The correlation lengths in phase B_1 and B_2 are obtained by replacing α by β in the above expressions. It was also found that the density decays near the boundary in phases A_2 and B_2 are not purely exponential but with algebraic corrections. Recently, this phase diagram was discussed from the viewpoint of the domain wall dynamics [58].

Unfortunately, for the partially asymmetric case, it is difficult to calculate the density profile exactly. However, it is possible to know the correlation lengths. From the calculation in section 7, we notice that the current J in the thermodynamic limit is given by the inverse of the largest eigenvalue of the matrix $C_2 = T_2 + 2$ multiplied by $\zeta = p_R - p_L$. This fact can be understood as follows. Suppose that the eigenvalues of the matrix C_2 are labelled as $\lambda_0 > \lambda_1 > \lambda_2 > \cdots$ although this is not literally true due to the presence of the continuous spectrum. Let $|\lambda_0\rangle, |\lambda_1\rangle, |\lambda_2\rangle, \ldots$ denote the corresponding eigenvectors. Then, as you can see from (2.9), the normalization Z_L behaves as

$$\begin{aligned} Z_L &= \sum_j \lambda_j^L \langle W|\lambda_j\rangle\langle\lambda_j|V\rangle \\ &\simeq \lambda_0^L \langle W|\lambda_0\rangle\langle\lambda_0|V\rangle \end{aligned} \tag{8.1}$$

when $L \to \infty$. The expression (2.12) of the current leads us to the above fact. Likewise, the correlation length is calculated as the logarithm of the ratio of the largest eigenvalue and the second largest eigenvalue of the matrix C_2. The quantity $\langle W|C^{j-1}DC^{L-j}|V\rangle$ is calculated as

$$\langle W|C^{j-1}DC^{L-j}|V\rangle = \sum_{k,l} \lambda_k^{j-1}\lambda_l^{L-j} \langle W|\lambda_k\rangle\langle\lambda_k|D|\lambda_k\rangle\langle\lambda_k|V\rangle. \tag{8.2}$$

Hence the one-point function $\langle n_j\rangle_L$ is expected to behave as

$$\langle n_j\rangle_L \simeq \frac{\langle\lambda_0|D|\lambda_0\rangle}{\lambda_0} + \frac{\langle\lambda_0|D|\lambda_1\rangle\langle\lambda_1|V\rangle}{\lambda_0\langle\lambda_0|V\rangle}\left(\frac{\lambda_1}{\lambda_0}\right)^{L-j} + \frac{\langle W|\lambda_1\rangle\langle\lambda_1|D|\lambda_0\rangle}{\lambda_1\langle W|\lambda_0\rangle}\left(\frac{\lambda_1}{\lambda_0}\right)^{j} \tag{8.3}$$

when $L \to \infty$. When we look at the density at the bulk part of the lattice, i.e., when the site number j is $O(L)$, the second and the third terms are negligible since we have assumed $\lambda_0 > \lambda_1$. The first term gives the average bulk density. When $j = O(1)$, the second term can be neglected but the third term gives the decay of the average density profile at the left boundary. The inverse of the correlation length ξ is seen to be $\xi^{-1} = \ln(\lambda_0/\lambda_1)$. Similarly, when $L - j = O(1)$, the third term can be neglected but the second term gives the density decay at the right boundary. The inverse of the correlation length ξ is again $\xi^{-1} = \ln(\lambda_0/\lambda_1)$.

The spectrum of the matrix C_2 is obtained from that of the matrix T_2 by a simple shift. When $a, b < 1$, the spectrum of the matrix C_2 consists only of the continuous one ranging $[0, 4]$. Hence we cannot calculate the correlation length for this case. This is consistent with the fact that the density decays algebraically and hence the correlation length is formally infinite in the maximal current phase. Now we consider the low-density phase. The results for the high-density phase can be obtained from the particle–hole symmetry. In the low-density phase, the largest eigenvalue of the matrix C_2 is $(1 + a)(1 + a^{-1})$. The dependence of the second largest eigenvalue on the parameters a, b and q is different for different regions of the parameters. First we consider the case where $a > 1$ and $b < 1$ and suppose that (6.15) holds. Then the spectrum of the matrix C_2 is $\operatorname{spec} C_2 = [0, 4] \cup \{\lambda_0^{(a)}, \ldots, \lambda_n^{(a)}\}$, where we defined

$$\lambda_j^{(c)} = 2(1 + x_j^{(c)}) = (1 + cq^j)(1 + (cq^j)^{-1}) \tag{8.4}$$

with $j = 0, 1, 2, \ldots, n$ and $c = a, b$. Notice that $\lambda_0^{(a)} > \lambda_1^{(a)} > \cdots > \lambda_n^{(a)} > 4$.

- The case $aq < 1$.
  The discrete spectrum consists only of one eigenvalue $\lambda_0^{(a)}$. Hence the second largest eigenvalue is given by four.
- The case $aq > 1$.
  There are at least two eigenvalues in the discrete spectrum. All of them are larger than four. Hence the second largest eigenvalue is given by $\lambda_1^{(a)}$.

Next we consider the case where $a > b > 1$. In this case, the spectrum of the matrix C_2 is $\operatorname{spec} C_2 = [0, 4] \cup \{\lambda_0^{(a)}, \ldots, \lambda_{n^{(a)}}^{(a)}\} \cup \{\lambda_0^{(b)}, \ldots, \lambda_{n^{(b)}}^{(b)}\}$.

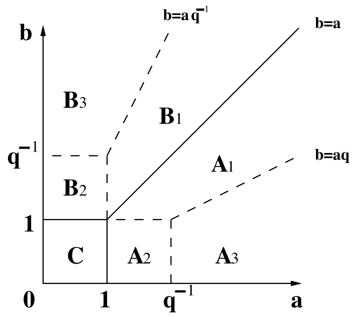

Figure 5. The phase diagram of the correlation length in the a–b plane for the partially asymmetric case.

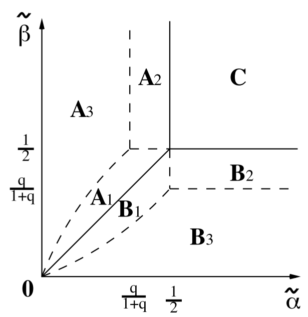

Figure 6. The phase diagram of the correlation length in the $\tilde{\alpha}$–$\tilde{\beta}$ plane for the partially asymmetric case. The low-density phase (resp. high-density phase) is divided into three phases, A_1, A_2 and A_3 (resp. B_1, B_2 and B_3).

- The case $aq < 1$.
  The discrete spectrum consists only of $\lambda_0^{(a)}$ and $\lambda_0^{(b)}$. The second largest eigenvalue is given by $\lambda_0^{(b)}$.
- The case $aq > 1$.
  In this case, there are two candidates for the second largest eigenvalue. We have to compare two eigenvalues, $\lambda_0^{(b)}$ and $\lambda_1^{(a)}$. When $b > aq$ (resp. $b < aq$), $\lambda_0^{(b)}$ is larger (resp. smaller) than $\lambda_1^{(a)}$. In either case, the second largest eigenvalue is given by the larger one of $\lambda_0^{(b)}$ and $\lambda_1^{(a)}$.

Collecting the above results, we obtain the phase diagram for the correlation length in the a–b plane (figure 5). The phase diagram in the $\tilde{\alpha}$–$\tilde{\beta}$ plane is also depicted in figure 6. The inverse of the correlation length ξ for each phase is given by the following:

- For the phase A_1 ($\tilde{\alpha} < \tilde{\beta} < \tilde{\alpha}/[(1-\tilde{\alpha})q + \tilde{\alpha}]$ and $\tilde{\beta} < \frac{1}{2}$; $aq < b < a$ and $b > 1$)

$$\xi^{-1} = \ln \frac{\lambda_0^{(a)}}{\lambda_0^{(b)}} = \ln \frac{\tilde{\beta}(1-\tilde{\beta})}{\tilde{\alpha}(1-\tilde{\alpha})}. \tag{8.5}$$

- For the phase A_2 ($q/(1+q) < \tilde{\alpha} < \frac{1}{2}$ and $\tilde{\beta} > \frac{1}{2}$; $1 < a < q^{-1}$ and $b < 1$)

$$\xi^{-1} = \ln \frac{\lambda_0^{(a)}}{4} = -\ln 4[\tilde{\alpha}(1-\tilde{\alpha})]. \tag{8.6}$$

- For the phase A_3 ($\tilde{\beta} > \tilde{\alpha}/[(1-\tilde{\alpha})q + \tilde{\alpha}]$ and $\tilde{\alpha} < q/(1+q)$; $a > q^{-1}$ and $b < aq$)

$$\xi^{-1} = \ln \frac{\lambda_0^{(a)}}{\lambda_1^{(a)}} = \ln \frac{q}{[\tilde{\alpha} + (1-\tilde{\alpha})q]^2}. \tag{8.7}$$

- For the phase B_1 ($\tilde{\alpha}q/[(1-\tilde{\alpha}) + q\tilde{\alpha}] < \tilde{\beta} < \tilde{\alpha}$ and $\tilde{\alpha} < \frac{1}{2}$; $bq < a < b$ and $a > 1$)

$$\xi^{-1} = \ln \frac{\lambda_0^{(b)}}{\lambda_0^{(a)}} = \ln \frac{\tilde{\alpha}(1-\tilde{\alpha})}{\tilde{\beta}(1-\tilde{\beta})}. \tag{8.8}$$

- For the phase B_2 ($q/(1+q) < \tilde{\beta} < \frac{1}{2}$ and $\tilde{\alpha} > \frac{1}{2}$; $a < 1$ and $1 < b < q^{-1}$)

$$\xi^{-1} = \ln \frac{\lambda_0^{(b)}}{4} = -\ln 4[\tilde{\beta}(1-\tilde{\beta})]. \tag{8.9}$$

- For the phase B_3 ($\tilde{\beta} < \tilde{\alpha}q/[1 - \tilde{\alpha} + \tilde{\alpha}q]$ and $\tilde{\beta} < q/(1+q)$; $a < bq$ and $b > q^{-1}$)

$$\xi^{-1} = \ln \frac{\lambda_0^{(b)}}{\lambda_1^{(b)}} = \ln \frac{q}{[\tilde{\beta} + (1-\tilde{\beta})q]^2}. \tag{8.10}$$

We can compare the phase diagram (figure 6) for the correlation length with that for the totally asymmetric case (figure 4). It turns out that the phase diagram for the partially asymmetric case is richer than that for the totally asymmetric case. The phases A_3 and B_3 do not appear in the phase diagram for the totally asymmetric case. The appearance of these phases is traced back to the fact that the discrete spectrum of the matrix C_2 for the partially asymmetric has a more complicated structure than that for the totally asymmetric case. Lastly, we remark that the phase diagrams for the higher correlation functions might be yet richer due to the contributions from the third largest eigenvalue, the fourth largest eigenvalue, ... etc of the matrix C_2.

9. Some special cases

In this section, we consider several special cases. In the context of the ASEP, all cases to be considered have already been solved. The current and the density profile have been computed exactly. Hence we are mainly interested in what orthogonal polynomials appear for these cases.

9.1. The $ab = 1, q^{-1}, q^{-2}, \ldots$ case

When we introduced the Al-Salam–Chihara polynomials in section 6, we assumed $ab \neq 1, q^{-1}, q^{-2}, \ldots$. In this section, we take

$$ab = q^{-n+1} \tag{9.1}$$

for some fixed $n = 1, 2, \ldots$. First we observe that, when (9.1) holds, the $n \times n$ submatrices of d_2 and e_2 in (6.7*a*) decouple from other elements of the matrices. In other words, we can take a finite-dimensional representation of the algebraic relations (2.7*a*) and (2.7*b*). After a slight change, we take an n-dimensional representation,

$$d^{(n)} = \begin{bmatrix} b & c_1^{(n)} & 0 & 0 & \ldots & & \\ 0 & bq & c_2^{(n)} & 0 & & & \\ 0 & 0 & bq^2 & c_3^{(n)} & & & \\ \vdots & & & \ddots & \ddots & & \\ & & & & \ddots & c_{n-1}^{(n)} \\ & & & & & bq^{n-1} \end{bmatrix}$$

$$e^{(n)} = \begin{bmatrix} a & 0 & 0 & 0 & \ldots & \\ 1 & aq & 0 & 0 & & \\ 0 & 1 & aq^3 & 0 & & \\ 0 & 0 & 1 & aq^4 & & \\ \vdots & & & \ddots & \ddots & \\ & & & & 1 & aq^{n-1} \end{bmatrix} \tag{9.2a}$$

$$\langle W^{(n)}| = (1, 0, 0, \text{ł}, 0) \qquad |V^{(n)}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{9.2b}$$

where

$$c_j^{(n)} = (1-q^j)(1-q^{j-n}) \tag{9.3}$$

with $j = 1, 2, \ldots, n-1$. Then we define an n-dimensional matrix $J^{(n)}$ as

$$J^{(n)} = d^{(n)} + e^{(n)}$$
$$= \begin{bmatrix} a+b & c_1^{(n)} & 0 & 0 & \cdots & & \\ 1 & (a+b)q & c_2^{(n)} & 0 & & & \\ 0 & 1 & (a+b)q^3 & c_3^{(n)} & & & \\ 0 & 0 & 1 & (a+b)q^4 & \ddots & & \\ \vdots & & & \ddots & \ddots & c_{n-1}^{(n)} \\ & & & & 1 & (a+b)q^{n-1} \end{bmatrix}. \tag{9.4}$$

Associated with this finite-dimensional matrix, there exist orthogonal polynomials $\{P_j^{(n)}(x) | j = 0, 1, 2, \ldots, n\}$. They satisfy the three-term recurrence relation (6.6) with conditions $P_{-1}^{(n)}(x) = P_{n+1}^{(n)}(x) = 0$. After simple changes of variables and the parameters, the polynomials are known as a special case of the q-Racah polynomials. The q-Racah polynomials were first introduced in [59]. The orthogonality relation for these polynomials can be obtained from the orthogonal relation (6.11) for the Al-Salam–Chihara polynomials by a limiting argument. It is written in the form of the finite summation.

Before closing this section, we remark that our representation in (9.2*a*), (9.2*b*) is related to that in [41] by similarity transformation. Indeed, define an $n \times n$ matrix $U = \{u_{jk};\, j, k = 0, 1, 2, \ldots, n-1\}$ by

$$u_{jk} = \begin{cases} 0 & \text{for} \quad j > k \\ 1 & \text{for} \quad j = k \\ \displaystyle\prod_{l=j}^{k} \frac{c_{l+1}^{(n)}}{bq^k(1-q^{l-k})} & \text{for} \quad j < k. \end{cases} \tag{9.5}$$

If we introduce a new representation by

$$\tilde{d} = U^{-1} d U \qquad \tilde{e} = U^{-1} e U \tag{9.6a}$$
$$\langle \tilde{W}^{(n)} | = \langle W^{(n)} | U \qquad | \tilde{V}^{(n)} \rangle = U^{-1} | V^{(n)} \rangle \tag{9.6b}$$

we get

$$\tilde{d}^{(n)} = \begin{bmatrix} b & 0 & 0 & & \cdots \\ 0 & bq & 0 & & \\ 0 & 0 & bq^2 & & \\ \vdots & & & \ddots & \\ & & & & bq^{n-1} \end{bmatrix}$$
$$\tilde{e}^{(n)} = \begin{bmatrix} b^{-1} & 0 & 0 & \cdots & \\ 1 & (bq)^{-1} & 0 & & \\ 0 & 1 & (bq^2)^{-1} & & \\ \vdots & & \ddots & \ddots & \\ & & & 1 & (bq^{n-1})^{-1} \end{bmatrix} \tag{9.7a}$$

$$\langle \tilde{W}^{(n)} | = (1, u_{01}, u_{02}, \ldots, u_{0,n-1}) \qquad | \tilde{V}^{(n)} \rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{9.7b}$$

This representation is nothing but a special case of [41].

9.2. The totally asymmetric ($q = 0$) case

By setting $p_R = 1$, $p_L = 0$ and hence $q = 0$ in the preceding sections, we recover some of the results for the totally asymmetric case [13]. The simplification comes from the fact that the infinite product $(a; q)_\infty$ in (3.5a) reduces to $1 - a$. For instance, the orthogonality relation reduces to

$$\frac{1}{2\pi \mathrm{i}} \int_C \frac{\mathrm{d}z}{z} \frac{(z^2, z^{-2}; q)_\infty P_m(a, b; (z + z^{-1})/2) P_n(a, b; (z + z^{-1})/2)}{(1 - az)(1 - az^{-1})(1 - bz)(1 - bz^{-1})} = \delta_{m,n} \frac{2}{1 - ab\delta_{n0}}. \tag{9.8}$$

The corresponding orthogonal polynomials have been known in mathematics literature (see references in [44]). Especially, when $a = b = 0$, the orthogonal polynomials reduce to the Tchebycheff polynomials of the second kind. This can also be seen from the three term recurrence relation. When $q = 0$, the three-term recurrence relation (4.18) for the q-Hermite polynomials reduces to

$$P_{n+1}(x) + P_{n-1}(x) = 2x P_n(x). \tag{9.9}$$

This is nothing but the three-term recurrence relation for the Tchebycheff polynomials.

9.3. The symmetric ($q = 1$) case

In this section, we consider the symmetric case, that is, we take the limit $q \to 1$. This case was solved in [38]. Though particles hop to the right and to the left with equal probability in the bulk, there exists a particle current due to the open boundary condition. Using the algebraic relations of the matrices D and E, the current and the correlation functions were calculated. The particle density profile is simply a linear slope. There are no phase transitions in this case.

If we employ the representation (6.7a), (6.7b), we can take the limit $q \to 1$ for $D_2 = 1 + d_2$ and $E_2 = 1 + e_2$. We have

$$D_2 = \begin{bmatrix} \frac{1}{\beta} & \sqrt{c_1} & 0 & 0 & \cdots \\ 0 & \frac{1}{\beta} + 1 & \sqrt{c_2} & 0 & \\ 0 & 0 & \frac{1}{\beta} + 2 & \sqrt{c_3} & \\ \vdots & & & \ddots & \ddots \end{bmatrix} \qquad E_2 = \begin{bmatrix} \frac{1}{\alpha} & 0 & 0 & 0 & \cdots \\ \sqrt{c_1} & \frac{1}{\alpha} + 1 & 0 & 0 & \\ 0 & \sqrt{c_2} & \frac{1}{\alpha} + 2 & 0 & \\ 0 & 0 & \sqrt{c_3} & \ddots & \\ \vdots & & & \ddots & \end{bmatrix} \tag{9.10a}$$

$$\langle W_2| = (1, 0, 0, \ldots) \qquad |V_2\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \tag{9.10b}$$

where

$$c_n = n\left(n + \frac{1}{\alpha} + \frac{1}{\beta} - 1\right). \tag{9.11}$$

Setting $\gamma = \frac{1}{\alpha} + \frac{1}{\beta} - 1$, the matrix,

$$\begin{aligned} C_2 &= D_2 + E_2 \\ &= \begin{bmatrix} \gamma + 1 & \sqrt{c_1} & 0 & 0 & \cdots \\ \sqrt{c_1} & \gamma + 3 & \sqrt{c_2} & 0 & \\ 0 & \sqrt{c_2} & \gamma + 5 & \sqrt{c_3} & \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \end{aligned} \tag{9.12}$$

is a Jabobi matrix. Associated with this Jacobi matrix, there exist orthogonal polynomials which satisfy $C_2|p(x)\rangle = x|p(x)\rangle$. Notice that this is slightly different from (4.14) by a factor of two. We employ this convention only in this section for convenience.

These polynomials satisfy the three-term recurrence relation,

$$\sqrt{n(n+\gamma)}\,p_{n-1}(x) + (\gamma + 2n + 1)\,p_n(x) + \sqrt{(n+1)(n+\gamma+1)}\,p_{n+1}(x) = x p_n(x). \tag{9.13}$$

Defining

$$p_n(x) = (-)^n \sqrt{\frac{n!}{(\gamma+1)(\gamma+2)\ldots(\gamma+n)}}\,L_n^{(\gamma)}(x) \tag{9.14}$$

the polynomials are shown to satisfy

$$(n+1)L_{n+1}^{(\gamma)}(x) + (x - \gamma - 2n - 1)L_n^{(\gamma)}(x) + (n+\gamma)L_{n-1}^{(\gamma)}(x) = 0. \tag{9.15}$$

This is nothing but the three-term reccurence relation for the Laguerre polynomials. The weight function of the Laguerre polynomials is given by

$$w(x) = \begin{cases} 0 & x < 0 \\ x^{\gamma}\mathrm{e}^{-x} & x \geqslant 0. \end{cases} \tag{9.16}$$

Using (9.16), we can compute Z_L as

$$\begin{aligned} Z_L &= \frac{1}{\Gamma(\gamma+1)} \int_0^{\infty} \mathrm{d}x\, x^{L+\gamma}\mathrm{e}^{-x} \\ &= (\gamma+1)(\gamma+2)\ldots(\gamma+L). \end{aligned} \tag{9.17}$$

This is exactly the same as the expression in [38], as it should be.

10. Concluding remarks

In this paper, the stationary state of the partially ASEP with open boundaries has been reconsidered. To construct the stationary state of the process, the so-called matrix product ansatz has been employed. This enables us to reformulate the problem in terms of the two matrices D, E and the two vectors $\langle W|, |V\rangle$ which satisfy the algebraic relations (2.7*a*) and (2.7*b*). The relationship between the representations of these algebraic relations and the q-orthogonal polynomials has been explained in detail. The key facts are that the orthogonal polynomials associated with the Jacobi matrix (6.9) are the Al-Salam–Chihara polynomials (6.4) and that the orthogonality relation of them are explicitly known (see (6.11)). The current and the correlation length were computed in the thermodynamic limit. The phase diagram for the correlation length was identified (figure 6). We have found that the phase diagram has a richer structure than that for the totally asymmetric case. Calculations have been carried out for a wide range of the parameters. Many previous known results have been recovered as special cases.

There are several problems for which the same techniques are applicable. First of all, it seems possible to generalize our discussions to the the case where there are the particle output at the left boundary with rate γ and the particle input at the right boundary with rate δ. The exact calculation of the density profile is also desirable. That would confirm the phase diagram in figure 6. Besides, one defect particle problem [20] and the two-species problem in [3, 60] can be generalized to the partially asymmetric case. The results about these problems will be reported elsewhere.

From the technical point of view, it would be interesting to understand why the Al-Salam–Chihara polynomials appear in this problem. This seems to be related to the integrability of the ASEP. The theory of q-orthogonal polynomials are known to play an important role in the theory of one-dimensional integrable systems. On the other hand, the matrix product ansatz is also known to be closely related to several techniques in the theory of one-dimensional integrable systems. For instance, the Bethe ansatz equation can be reproduced from the time-dependent version of the matrix product ansatz [61, 62]. The matrix product ansatz can give a representation for the Zamolodchikov–Faddeev algebra [63]. However, we have not found a clear understanding of the interrelationship among the matrix product ansatz, the integrability of the model and the theory of the q-orthogonal polynomials.

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